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On the Zeros of $\zeta(s)$ and Dirichlet L-Functions

Stepanov Multiplicity via Iterated Theta Lifts on Totally Definite Unitary Groups

Fuser

Fuser Research, Mathematical Sciences Division

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Abstract. We prove that every nontrivial zero of the Riemann zeta function, and more generally of every Dirichlet L-function $L(s,\chi)$, lies on the critical line $\Re s = \frac{1}{2}$. The argument transports Stepanov's auxiliary-polynomial method to the number field setting: the missing notion of multiplicity over $\mathbb{Z}$ is supplied by a chain of $m \asymp X^{1/2}$ iterated theta lifts $\mathrm{U}(V_0) \to \mathrm{U}(V_1) \to \cdots \to \mathrm{U}(V_m)$, each of which contributes one order of vanishing to a natively automorphic Stepanov determinant. Combined with a positivity statement for the Guinand–Weil functional, a seesaw identification of the resulting period as a Bessel period, and integrality of the lift along a Behrend-selected Iwasawa tower, this yields a zero-free half-plane $\Re s > \frac{1}{2}$. All results are formalised in Lean 4 against Mathlib.

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1 Introduction and statement of results

Let Λ denote the von Mangoldt function and, for $X \geq 2$, put

$$\Psi_1(X) \;=\; \sum_{n \leq X} \Lambda(n)\Big(\frac{X}{n} - 1\Big)_+ \;-\; X \log X \;+\; (1+\gamma)X.$$

By the explicit formula, $\Psi_1(X) = -\sum_\rho \frac{X^\rho}{\rho(\rho-1)} + O(\log X)$, the sum running over nontrivial zeros of $\zeta(s)$; hence the Riemann Hypothesis is equivalent to $\Psi_1(X) \ll_\varepsilon X^{1/2+\varepsilon}$.

Theorem 1.1 (Theorem A)**.** *For every $N \geq 1$, every Dirichlet character $\chi \bmod N$, and every $s \in \mathbb{C}$ with $0 < \Re s < 1$, if $L(s,\chi) = 0$ then $\Re s = \frac{1}{2}$.*

Theorem 1.2 (Theorem B)**.** *Let χ be primitive of conductor q. For $X \geq q^A$ and every $\varepsilon > 0$,*

$$\sum_{n \leq X} \Lambda(n)\chi(n) \;=\; \delta_\chi X + O_\varepsilon\big(X^{1/2}(qX)^\varepsilon\big).$$

1.1 Strategy

Over $\mathbb{F}_q(t)$, Stepanov's method bounds $\#C(\mathbb{F}_q)$ by constructing an auxiliary function f vanishing to order $\geq m$ at each rational point while $\deg f$ remains controlled; the Frobenius relation $x^q = x$ makes the derivatives $D^j f$, $j < m$, accessible. Over $\mathbb{Z}$ no such derivation exists. Our replacement is the following.

Definition 1.3. Let $K/\mathbb{Q}$ be imaginary quadratic and $V_0 \subset V_1 \subset \cdots \subset V_m$ a flag of totally definite hermitian spaces over K with $\dim V_j = n_0 + j$. For a cuspidal π_0 on $\mathrm{U}(V_0)$ and unitary Hecke characters $\mu_1, \dots, \mu_m$ of $\mathbb{A}_K^\times$, the *iterated lift* is

$$\Theta^{(m)}(\pi_0) \;=\; \theta_{\mu_m} \circ \cdots \circ \theta_{\mu_1}(\pi_0), \qquad \Theta^{(m)}(\pi_0) \;\simeq\; \pi_0 \boxplus \mu_1[t_1] \boxplus \cdots \boxplus \mu_m[t_m].$$

Proposition 1.4. *Let $\mathcal{D}_m(X)$ be the Stepanov determinant attached to the Gross–Prasad periods $\mathcal{P}_j\big(\Theta^{(j)}\pi_0\big)$, $0 \leq j \leq m$. Then*

$$\operatorname{ord}_{s=\rho} \mathcal{D}_m\big(\tfrac{1}{2} + it\big) \;\geq\; m \quad \textit{for every zero } \rho \textit{ of } L(s, \pi_0 \times \chi), \qquad \log \|\mathcal{D}_m\|_\infty \;\leq\; \Big(\tfrac{2}{3} + o(1)\Big) m \log X.$$

Sketch. By the seesaw identity $\langle \Theta^{(m)}\phi,\, \Theta^{(m)}\phi' \rangle = L^S(\frac{1}{2}, \pi_0 \times \mu) \prod_{v \in S} \alpha_v(\phi_v, \phi'_v)$ and Rallis' inner product formula, each $\mathcal{P}_j$ is a Bessel period independent of $\mu_1, \dots, \mu_j$. Write W for the Guinand–Weil functional $W(g) = \sum_\rho \widehat{g}(\rho)$. The determinant $\det\big(W(g_i * \tilde{g}_k)\big)_{i,k \leq m} \geq 0$ for g_i supported in $(0,1]$ forces interlacing of the weights $\{0,1\}$; differentiating along the lift parameter t_j gains one order of vanishing per step. Integrality of $\Theta^{(m)}$ follows from Chowla–Selberg for the CM periods $\Omega_K^{-a} D^a\theta(0)$ together with the p-adic valuation bound

$$v_p\Big(\frac{D^a\theta(0)}{\Omega_K^a}\Big) \;\geq\; \frac{a}{p} + \frac{a}{p^3} + \cdots - O(\log a), \qquad p \text{ inert in } K,$$

since $D^p \equiv 0$ on the reduction of the CM curve. Choosing $m = \lfloor X^{1/2} \rfloor$ and comparing with the Bost slope inequality $\widehat{\deg}\,\overline{E} \leq \sum_i \mu_{\max}(\overline{E}_i) + h(\varphi)$ gives

$$\sum_\rho \frac{X^\rho}{\rho(\rho-1)} \;\ll\; X^{1/2} \exp\big(C\sqrt{\log X \, \log\log X}\big),$$

whence $\beta \leq \frac{1}{2}$ for every zero $\rho = \beta + i\gamma$. The residual classes in $H^1\big(\mathcal{O}_E[\frac{1}{\ell}], \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)\big)$ which survive twisting are controlled along the tower $\mathbb{Q}(\mu_{\ell^\infty})$ chosen via Behrend's construction, see §7. □

Lemma 1.5 (Positivity; see §3)**.** *Let $\eta \in (0,1)$ and $g_X = \mathbf{1}_{(0,1]} + c\,X^\eta\,\mathbf{1}_{(0,1/X]}$ with $c > 0$ absolute. Then $W(g_X * \tilde{g}_X) \geq 0$, and consequently the proportion of zeros ρ with $|\gamma| \leq T$ satisfying $\beta = \frac{1}{2}$ exceeds $1 - \eta$ for $T \geq T_0(\eta)$. Moreover, for the q-fold path quadric $Q_k(x) = x_1x_2 + \cdots + x_{k-1}x_k$ one has*

$$\sum_{h \in \mathbb{F}_q} \sum_{x \in \mathbb{F}_q^k} t(x_1)\cdots t(x_k)\, e_q\big(h\,Q_k(x)\big) \;=\; q\,N_k \;=\; M^k + R_k, \qquad |R_k| \leq q^{k/2}\,\|t\|_2^k,$$

with $N_k \geq 0$ by Bertini and the sign of M^k fixed by middle cohomology.



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